Exercise 15

Evaluate the line integral, where C is the given curve.

$$\int_C z^2\,dx + x^2\,dy + y^2\,dz, \quad C \text{ is the line segment from } (1,0,0) \text{ to } (4,1,2)$$

Solution

The equation of the line going from $(1,0,0)$ to $(4,1,2)$ is

$$\begin{aligned} \mathbf{y} &= \mathbf{m}t + \mathbf{b} \\ &= \langle 4-1, 1-0, 2-0\rangle t + \langle 1,0,0\rangle \\ &= \langle 3t, t, 2t\rangle + \langle 1,0,0\rangle \\ &= \langle 3t+1, t, 2t\rangle. \end{aligned}$$

Write the integral in terms of a dot product.

$$\int_C z^2\,dx + x^2\,dy + y^2\,dz = \int_C \langle z^2, x^2, y^2\rangle \cdot \langle dx, dy, dz\rangle$$

With the parameterization $x(t) = 3t+1$ and $y = t$ and $z = 2t$, where $0 \le t \le 1$, the line integral becomes

$$\begin{aligned} \int_C z^2\,dx + x^2\,dy + y^2\,dz &= \int_0^1 \langle [z(t)]^2, [x(t)]^2, [y(t)]^2\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\rangle dt \\ &= \int_0^1 \langle (2t)^2, (3t+1)^2, (t)^2\rangle \cdot \langle 3,1,2\rangle\,dt \\ &= \int_0^1 \langle 4t^2, (3t+1)^2, t^2\rangle \cdot \langle 3,1,2\rangle\,dt \\ &= \int_0^1 [4t^2(3) + (3t+1)^2(1) + t^2(2)]\,dt \\ &= \int_0^1 (1 + 6t + 23t^2)\,dt \\ &= \left(t + 3t^2 + \frac{23}{3}t^3\right)\bigg|_0^1 \\ &= 1 + 3 + \frac{23}{3} \\ &= \frac{35}{3}. \end{aligned}$$